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# A note on Whitham's Rule

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Whitham (1957, 1958) has developed a method for treating the propagation of shock waves into non-uniform media, which has been widely and successfully used. In shock diffraction problems, he considers the medium ahead of the shock to be at rest. The purpose of this note is to generalize Whitham's technique to problems having a uniform flow ahead of the shock, in which the inclination of the shock normal to the direction of flow ahead of the shock remains small. This extension was suggested to the author by Miles (1964) who considered the problem of diffraction of a strong shock by a thin supersonic wedge.

The fluid ahead of the shock has Mach number $M$ and sound speed $a_2$. The shock has a Mach number $\mathscr{M}$, relative to the flow ahead. Following Whitham we introduce orthogonal co-ordinates $(\alpha, \beta)$, positions of the shock being given by $\alpha =$ const. and the shock normals, or rays, by $\beta =$ const. The point P of figure 1 has co-ordinates $(\alpha, \beta)$, a ray element PQ $= \delta s = (\mathscr{M} + M\cos\theta)\,\delta\alpha$ and

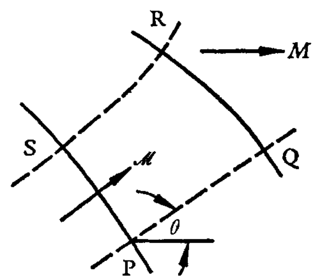


FIGURE 1. The shock-front PS moves with Mach number $\mathscr{M}$ into a uniform flow having Mach number $M$. QR is a later position of the shock and PQ, RS are the orthogonal trajectories, or rays, inclined at $\theta$ to the uniform flow.

an element of the shock front PS $= A\delta\beta$, $\theta$ being the ray inclination to the flow ahead of the shock. There are two relations which follow from P, Q, R, S being an orthogonal quadrangle,

$$\frac{\partial\theta}{\partial\beta} = \frac{1}{(\mathscr{M} + M\cos\theta)}\frac{\partial A}{\partial\alpha}, \quad \frac{\partial\theta}{\partial\alpha} = -\frac{1}{A}\frac{\partial}{\partial\beta}(\mathscr{M} + M\cos\theta). \tag{1}$$

For the case $M = 0$, Whitham (1957) completed the theory by incorporating an assumed relation $A(\mathscr{M})$ deduced from studies of shocks in channels. He assumed that the rays may be considered as channel walls, although they are known to be so only immediately behind the shock front. In 1958 he showed that by applying the one-dimensional channel flow characteristic relation

$$dp + \rho a du + \rho a^2 u A^{-1}(u+a)^{-1} dA = 0 \tag{2}$$

to the flow variables behind the shock, the same $A(\mathscr{M})$ relation is recovered.

Chester (1960) extended this result to steady channel flow ahead of the shock. In the work of Whitham and Chester the fluid ahead of the shock has no velocity component tangential to the shock front. By considering in this note a uniform flow ahead of the shock, we introduce into the problem a fluid velocity component tangential to the shock. This tangential velocity component is preserved across the shock, and the rays cannot, therefore, be considered as channel walls, even immediately behind the shock. As $\theta$ is assumed small, the tangential fluid velocity component is small and we show that the fluid equations for problems containing a small velocity component tangential to the shock may be reduced to the corresponding equations for problems in which the medium ahead of the shock is at rest, but with the ray tubes appropriately elongated.

This result is deduced from a study of the continuity equation. The mass rate of outflow from the quadrangle P, Q, R, S, across the sides PQ and RS is

$$\begin{aligned} -\frac{\partial}{\partial\beta}\{\rho M \sin\theta a_2(\mathscr{M}+M\cos\theta)\,\delta\alpha\}\,\delta\beta \\ &= -\rho M a_2(\partial\theta/\partial\beta)\,(\mathscr{M}+M)\,\delta\alpha\,\delta\beta, \quad \text{for small } \theta, \\ &= -\rho M a_2(\partial A/\partial\alpha)\,\delta\alpha\,\delta\beta, \quad \text{by (1)}, \\ &= -\rho M a_2(\partial A/\partial s)\,\delta s\,\delta\beta. \end{aligned}$$

The continuity equation for one-dimensional channel flow between rays may be modified to incorporate the effect of outflow across the rays, becoming

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial s}+\rho\left\{\frac{\partial u}{\partial s}+\frac{u}{A}\frac{\partial A}{\partial s}\left(1-\frac{Ma_2}{u}\right)\right\}=0.$$

The effect of the tangential fluid velocity component may be interpreted as a channel elongation, the term $(\rho u/A)\,\partial A/\partial s$ applicable in the case of no motion ahead of the shock being replaced by $\rho u A^{-1}(\partial A/\partial s)(1-Ma_2/u)$.

As the area function enters the channel equations only through the continuity equation, the Whitham characteristic rule (2) becomes

$$dp+\rho a\,du+\frac{\rho a^2 u}{(u+a)}\left(1-\frac{Ma_2}{u}\right)\frac{dA}{A}=0, \tag{3}$$

which is to be applied to the flow variables behind the shock. We note from the shock equation for $u$, that the factor $(1-Ma_2/u)$ may be written

$$\left\{1+\frac{(\gamma+1)\mathscr{M}M}{2(\mathscr{M}^2-1)}\right\}^{-1}.$$

It is anticipated that this correction factor will be applicable for large $\mathscr{M}$, but will be invalid as $\mathscr{M}\to 1$, for fixed $M$, as the flow behind the shock no longer approximates to channel flow between rays, but is a uniform flow.

This work was done while the author was a visitor to the Lockheed Missiles and Space Company, Palo Alto, California.

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